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Polynomial minorants, majorants for Sine, Cosine and Tangent Without Calculus

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ABSTRACT. This paper aim is consideration of elementary (calculus-free) way which only the double-angle formulas and the inequalities $\sin x < x < \tan x$ give opportunity to obtain well known polynomial minorants and majorants for $\sin x, \cos x$ and $\tan x$.

1 INTRODUCTION

We offer an elementary (calculus-free) proof of the inequalities

$$x - \frac{x^3}{6} < \sin x < x - \frac{x^3}{6} + \frac{x^5}{120}, \tag{1}$$

$$1 - \frac{x^2}{2} < \cos x < 1 - \frac{x^2}{2} + \frac{x^4}{24} \tag{2}$$

$$x + \frac{x^3}{3} < \tan x, \tag{3}$$

when $x \in \left(0, \frac{\pi}{2}\right)$. All we need are the double-angle formulas

$$\sin 2x = 2 \sin x \cos x,$$

$$\cos 2x = 1 - 2\sin^2 x,$$

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x},$$

all presented in standard courses of trigonometry, and the double inequality $\sin x < x < \tan x$ for $x \in (0, \pi/2)$, which has a visual geometric proof.

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The left side of(2) can be improved to become

$$1-\frac{x^2}{2}<1-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^6}{720}<\cos x \tag{4}$$

and we prove this as well. In each case the polynomials consist of the initial terms of Maclaurin series for the sine, cosine, and tangent functions. The left sides of (1)-(4) holds for all $x>0$.
When we say "calculus-free proof" we mean that proof should be free not only from derivative and series but also free from limits, using instead "passing to limit", the simple and transparent reasoning in form of following

Proposition 1. Let P be set of positive real numbers such that for any positive real ε there is $p\in P$ that $p<\varepsilon$ and let inequality $f(x)<p$ holds for any $x\in Dom(f)$ and any $p\in P$. Then for any $x\in Dom(f)$ holds inequality $f(x)\leq 0$.

proof. Indeed, supposition of existence $x_0\in Dom(f)$ such that $f(x_0)>0$ immediately leads to contradiction because then $f(x_0)<p$ for any $p\in P$ and at the same time for $\varepsilon=f(x_0)$ there is $p\in P$ that $p<f(x_0)$.

- Proof of $x-x^3/6<\sin x, x\in(0,\pi/2)$

For $x\in(0,\pi/2)$ the inequality $x<\tan x$ yields $x\cos x<\sin x$, so for any $x\in(0,\pi/2)$ we have

$$x\cos x<\sin x<x \tag{5}$$

Using (5) we can find a third-degree polynomial that is a lower bound for $\sin x$. Indeed, since $\cos x=1-2\sin^2(x/2)$ and $\sin(x/2)<x/2$ we have $\cos x>1-2\left(\frac{x}{2}\right)^2=1-\dfrac{x^2}{2}$ (which is the left side of (2)) and, therefore,

$$\sin x>x\cos x>x\left(1-\frac{x^2}{2}\right)=x-\frac{x^3}{2}.$$

Suppose now that for some positive a the inequality $\sin x\geq x-ax^3$ holds for every $x\in(0,\pi/2)$.(We have just shown that this holds when $a=1/2$). Then, since $\cos x>1-x^2/2$, we obtain

$$sinx=2\sin\frac{x}{2}\cos\frac{x}{2}>2\left(\frac{x}{2}-a\left(\frac{x}{2}\right)^3\right)\left(1-\frac{1}{2}\left(\frac{x}{2}\right)^2\right)=$$

$$= x - \left(\frac{a}{4} + \frac{1}{8}\right)x^3 + \frac{1}{32}ax^5 > x - \left(\frac{a}{4} + \frac{1}{8}\right)x^3$$

whenever $x \in (0, \pi)$ and, therefore, for any $x \in (0, \pi/2)$ holds inequality

$$\sin x > x - \left(\frac{a}{4} + \frac{1}{8}\right)x^3 \iff x - \sin x < \left(\frac{a}{4} + \frac{1}{8}\right)x^3$$

So, we can see that if for some $a > 0$ inequality $x - \sin x < ax^3$ holds for any $x \in (0, \pi/2)$ then inequality
$x - \sin x < \left(\frac{a}{4} + \frac{1}{8}\right)x^3$ holds for any $x \in (0, \pi/2)$ as well.
(Furthermore, even non-strict inequality $x - \sin x \le ax^3, x \in (0, \pi/2)$ by procedure represented above generate strict inequality
$x - \sin x < \left(\frac{a}{4} + \frac{1}{8}\right)x^3$ for $x \in (0, \pi)$).
Thus, for any term a_n of the sequence defined by
$a_{n+1} = \frac{a_n}{4} + \frac{1}{8}, a_1 = 1/2$ inequality $r(x) < a_n x^3$ holds for any $x \in (0, \pi/2)$.
Since $\frac{1}{8} = c - \frac{c}{4} \iff c = \frac{1}{6}$ then for any $n \in \mathbb{N}$ we have

$$a_{n+1} = \frac{a_n}{4} + \frac{1}{8} \iff a_{n+1} - \frac{1}{6} = \frac{1}{4}\left(a_n - \frac{1}{6}\right) \iff$$

$$\iff 4^{n+1}\left(a_{n+1} - \frac{1}{6}\right) = 4^n\left(a_n - \frac{1}{6}\right).$$

That is

$$4^n\left(a_n - \frac{1}{6}\right) = 4^1\left(a_1 - \frac{1}{6}\right) = 4\left(\frac{1}{2} - \frac{1}{6}\right) = \frac{4}{3}$$

and, therefore, $a_n = \frac{1}{6} + \frac{1}{3 \cdot 4^{n-1}}, n \in \mathbb{N}$.
Thus,

$$x - \sin x - \frac{x^3}{6} < \frac{x^3}{3 \cdot 4^{n-1}} < \frac{\pi^3}{6 \cdot 4^n}, n \in \mathbb{N}.$$

Since $\frac{1}{4^n} < \frac{1}{3n}$ for any $n \in \mathbb{N}$ (can be proved by Math Induction) then
$x - \sin x - \frac{x^3}{6} < \frac{\pi^3}{18n}, n \in \mathbb{N}$ and applying Proposition to

$f(x) = x - \sin x - \dfrac{x^3}{6}$ and $P = \left\{ \dfrac{\pi^3}{18n} \mid n \in \mathbb{N} \right\}$ we obtain inequality

$x - \sin x - \dfrac{x^3}{6} \le 0 \iff x - \sin x \le \dfrac{x^3}{6}$ for any $x \in (0, \pi/2)$.

Since $x - \sin x \le \dfrac{x^3}{6}$ yields $x - \sin x < \left(\dfrac{1}{6} \cdot \dfrac{1}{4} + \dfrac{1}{8} \right) x^3 = x - \dfrac{x^3}{6}$ we finally get strict inequality

$$x - \sin x < \frac{x^3}{6} \iff x - \frac{x^3}{6} < \sin x$$

for any $x \in (0, \pi/2)$.

- Proof of $\cos x < 1 - \dfrac{x^2}{2} + \dfrac{x^4}{24}, x \in (0, \pi/2)$.

Since $\sin x > x - \dfrac{x^3}{6} > 0$ $\left(x - \dfrac{x^3}{6} = \dfrac{x}{24}(24 - 4x^2) > \dfrac{x}{24}(24 - \pi^2) > 0\right)$, we have

$$\cos x = 1 - 2\sin^2\left(\frac{x}{2}\right) \le 1 - 2\left(\frac{x}{2} - \frac{1}{6}\left(\frac{x}{2}\right)^3\right)^2 = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{1152}x^6 <$$

$$< 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4.$$

- Proof of $\sin x < x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!}, x \in (0, \pi/2)$

Let $x \in (0, \pi/2)$. Since $\sin x < x$ and $\cos x < 1 - \dfrac{1}{2}x^2 + \dfrac{1}{24}x^4$ then

$\sin x = 2\sin\dfrac{x}{2}\cos\dfrac{x}{2} < 2 \cdot \dfrac{x}{2}\left(1 - \dfrac{(x/2)^2}{2} + \dfrac{(x/2)^4}{24}\right) = x - \dfrac{x^3}{8} + \dfrac{x^5}{384}$. So, we have

$$\sin x < x - \frac{x^3}{8} + \frac{x^5}{384}, \quad x \in (0, \pi/2) \tag{6}$$

Suppose now that for some positive a and b the inequality $\sin x \le x - ax^3 + bx^5$ holds for every $x \in (0, \pi/2)$. Then

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} < 2\left(\frac{x}{2} - a\left(\frac{x}{2}\right)^3 + b\left(\frac{x}{2}\right)^5\right)\left(1 - \frac{1}{2}\left(\frac{x}{2}\right)^2 + \frac{1}{24}\left(\frac{x}{2}\right)^4\right) =$$

$$x-\left(\frac{a}{4}+\frac{1}{8}\right)x^3+\left(\frac{a}{32}+\frac{b}{16}+\frac{1}{384}\right)x^5-\left(\frac{x}{2}\right)^7\left(\frac{1}{12}a+b\left(1-\frac{x^2}{48}\right)\right)<$$

$$<x-\left(\frac{a}{4}+\frac{1}{8}\right)x^3+\left(\frac{a}{32}+\frac{b}{16}+\frac{1}{384}\right)x^5$$

because $\frac{1}{12}a+b\left(1-\frac{x^2}{48}\right)>0$ for $x<\frac{\pi}{2}$.

Thus, the non-strict inequality $\sin x\leq x-ax^3+bx^5$ (which we know to be true when $a=1/8$ and $b=1/384$) yields the strict inequality

$$\sin x<x-\left(\frac{a}{4}+\frac{1}{8}\right)x^3+\left(\frac{a}{32}+\frac{b}{16}+\frac{1}{384}\right)x^5.$$

If we write $a=\frac{1}{6}-p,b=\frac{1}{120}-q$ and denote $\sin x-x+\frac{x^3}{6}-\frac{x^5}{120}$ via $r(x)$ then

$$\frac{a}{4}+\frac{1}{8}=\frac{1}{6}-\frac{p}{4},\ \frac{a}{32}+\frac{b}{16}+\frac{1}{384}=\frac{1}{32}\left(\frac{1}{6}-p\right)+\frac{1}{16}\left(\frac{1}{120}-q\right)+\frac{1}{384}=$$

$$=\frac{1}{120}-\frac{p}{32}-\frac{q}{16}$$

and

$$\sin x\leq x-ax^3+bx^5\iff r(x)\leq px^3+qx^5,$$

$$\sin x<x-\left(\frac{a}{4}+\frac{1}{8}\right)x^3+\left(\frac{a}{32}+\frac{b}{16}+\frac{1}{384}\right)x^5\iff$$

$$\iff r(x)<\frac{p}{4}x^3+\left(\frac{p}{32}+\frac{q}{16}\right)x^5.$$

So, we have shown that the inequality $r(x)<px^3+qx^5$ and even $r(x)\leq px^3+qx^5$ implies the inequality $r(x)<\frac{p}{4}x^3+\left(\frac{p}{32}+\frac{q}{16}\right)x$.

Due to inequality (6) with $a=\frac{1}{8}$ and $b=\frac{1}{384}$ the initial value of p is $\frac{1}{6}-\frac{1}{8}=\frac{1}{24}$ and the initial value of q is $\frac{1}{120}-\frac{1}{384}=\frac{11}{1920}$.

Note that $\frac{11}{1920}\div\frac{1}{24}=\frac{11}{80}<\frac{1}{6}$ and supposition $q<\frac{p}{6}$ implies

$$\frac{p}{32}+\frac{q}{16}\le\frac{p}{32}+\frac{1}{16}\cdot\frac{p}{6}=\frac{p}{24}.$$

Hence,

$$r(x)<px^3+qx^5<p\left(x^3+\frac{1}{6}x^5\right)$$

yields

$$r(x)<\frac{p}{4}x^3+\left(\frac{p}{32}+\frac{q}{16}\right)x^5<\frac{p}{4}\left(x^3+\frac{1}{6}x^5\right).$$

So, inequality $\sin x-x+\dfrac{x^3}{6}-\dfrac{x^5}{120}<p_n\left(x^3+\dfrac{1}{6}x^5\right)$ holds for $p_n, n\in\mathbb{N}$ defined by $p_1=1/24$ and $p_{n+1}=\dfrac{p_n}{4}, n\in\mathbb{N}$,that is it holds for $p_n=\dfrac{1}{24}\cdot\dfrac{1}{4^{n-1}}=\dfrac{1}{6\cdot4^n}, n\in\mathbb{N}$.
Since $\dfrac{1}{6\cdot4^n}<\dfrac{1}{18n}$ can be arbitrary small with increasing n then

$$\frac{1}{6\cdot4^n}\left(x^3+\frac{1}{6}x^5\right)<\frac{1}{6\cdot4^n}\left((\pi/2)^3+\frac{1}{6}(\pi/2)^5\right)$$

can be arbitrary small with increasing n as well and, therefore, applying Proposition to $f(x)=r(x)$ we obtain inequality

$$r(x)\le0, x\in(0,\pi/2)$$

which equivalent to inequality

$$\sin x\le x-\frac{1}{6}x^3+\frac{1}{120}x^5, x\in(0,\pi/2)$$

and, since $\sin x\le x-\dfrac{1}{6}x^3+\dfrac{1}{120}x^5$ yields

$$\sin x<x-\left(\frac{1}{6}\cdot\frac{1}{4}+\frac{1}{8}\right)x^3+\left(\frac{1}{6}\cdot\frac{1}{32}+\frac{1}{120}\cdot\frac{1}{16}+\frac{1}{384}\right)x^5=x-\frac{1}{6}x^3+\frac{1}{120}x^5$$

we finally get strict inequality

$$\sin x<x-\frac{1}{6}x^3+\frac{1}{120}x^5.$$

- Proof of $1-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^6}{720}<\cos x, x\in(0,\pi/2)$.

As a consequence of the above inequalities for $\sin x$ we have

$$\cos x = 1-2\sin^2\left(\frac{x}{2}\right) \geq 1-2\left(\frac{x}{2}-\frac{1}{3!}\left(\frac{x}{2}\right)^3+\frac{1}{5!}\left(\frac{x}{2}\right)^5\right)^2 =$$

$$=1-2\left(\frac{x^2}{4}-\frac{x^4}{3!2^3}+\frac{1}{45}\cdot\frac{x^6}{2^5}-\frac{1}{3!5!}\left(\frac{x}{2}\right)^8\left(2-\frac{1}{4\cdot 5}\left(\frac{x}{2}\right)^2\right)\right)=$$

$$=1-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^6}{720}+\frac{2}{3!5!}\left(\frac{x}{2}\right)^8\left(2-\frac{1}{4\cdot 5}\left(\frac{x}{2}\right)^2\right)>1-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^6}{720}$$

because $2\frac{1}{4\cdot 5}\left(\frac{x}{2}\right)^2<2$ for $x\in(0,\pi/2)$.

- Proof of $x+x^3/3<\tan x, x\in(0,\pi/2)$

Let $x\in(0,\pi/2)$ .Since $\tan x>x$ we obtain

$$\tan x=\frac{2\tan\left(\frac{x}{2}\right)}{1-\tan^2\left(\frac{x}{2}\right)}>\frac{2\cdot\frac{x}{2}}{1-\left(\frac{x}{2}\right)^2}>x\left(1+\frac{x^2}{4}\right)=x+\frac{x^3}{4}.$$

As above, suppose that for some $a>0$ inequality $\tan x\geq x+ax^3$ holds for any $x\in(0,\pi/2)$ .Then

$$\tan x=\frac{2\tan\left(\frac{x}{2}\right)}{1-\tan^2\left(\frac{x}{2}\right)}>\frac{2\left(\frac{x}{2}+\frac{ax^3}{8}\right)}{1-\left(\frac{x}{2}+\frac{ax^3}{8}\right)^2}>\left(x+\frac{ax^3}{4}\right)\left(1+\left(\frac{x}{2}+\frac{ax^3}{8}\right)^2\right)>$$

$$>\left(x+\frac{ax^3}{4}\right)\left(1+\frac{x^2}{4}\right)>x+\left(\frac{a}{4}+\frac{1}{4}\right)x^3.$$

Thus, the non strict inequality $\tan x\geq x+ax^3$ (which we know to be true when $a=1/4$) yields the strict inequality $\tan x>x+\left(\frac{a}{4}+\frac{1}{4}\right)x^3$.

If we write $a = \frac{1}{3} - p$,then $\frac{a}{4} + \frac{1}{4} = \frac{1}{3} - \frac{p}{4}$. So we have shown that the inequality $r(x) < px^3$ implies the inequality $r(x) < \frac{px^3}{4}$, where
$r(x) = x + \frac{x^3}{3} - \tan x$.
We know that the inequality holds for $p = 1/12$ because

$$\tan x > x + \frac{x^3}{4} \iff r(x) < \frac{x^3}{12}.$$

So, inequality $r(x) < px^3$ holds for
$p = \frac{1}{48}, \frac{1}{192}, \ldots, \frac{1}{12 \cdot 4^n}, \ldots$ in fact for arbitrary small positive p. Therefore, applying Proposition to the function $f(x) = r(x)$ we obtain
$r(x) \leq 0 \iff x + \frac{x^3}{3} \leq \tan x$ for $x \in (0, \pi/2)$ and since
$\tan x \geq x + \frac{1}{3}x^3$ yields

$$\tan x > x + \left(\frac{1}{3} \cdot \frac{1}{4} + \frac{1}{4}\right) x^3 = x + \frac{1}{3}x^3$$

we finally get strict inequality

$$\tan x > x + \frac{1}{3}x^3.$$

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